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Functional Relations in Stokes Multipliers—Fun with $x^6+\alpha x^2$ Potential[1]

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We consider eigenvalue problems in quantum mechanics in one dimension. Hamiltonians contain a class of double well potential terms, $x^6+\alpha x^2$, for example. The space coordinate is continued to a complex plane and the connection problem of fundamental system of solutions is considered. A hidden $U_q(\widehat{gl}(2\,|\,1))$ structure arises in "fusion relations" of Stokes multipliers. With this observation, we derive coupled nonlinear integral equations which characterize the spectral properties of both $\pm\alpha$ potentials simultaneously.

KEY WORDS: Spectral determinants; Stokes multipliers; fusion hierarchy; nonlinear integral equations.

1. INTRODUCTION

The eigenvalue problem of a one-body 1D Schrödinger operator is the most fundamental subject in quantum mechanics. Still, it provides vivid materials of research.

Besides a few exceptions where eigenvalues and wavefunctions are obtainable explicitly, one may employ several tools for analysis, e.g., the perturbation theory, the variational approach and so on. Among them, the exact WKB method$^{(1-8)}$ is unique in the sense that it provides non-perturbative information on the analytical structure of wavefunctions and spectral properties. We analytically continue x, original coordinate variable, to a complex number. The whole complex plane is divided into several sectors. In each sectors, there are two linearly independent solutions, as Schrödinger operator is the 2nd order differential operator. They are

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referred to as the fundamental system of solutions (FSS) in the sector.[18] The relations among FSS in different sectors are central issue in the connection problem. The importance of the problem and consequently Stokes multipliers, in the WKB problem has been deeply recognized and emphasized in early '80s, especially in ref. 1.

Recently, a remarkable link has been established among the spectral determinants of a 1D Schrödinger operator associated with the anharmonic oscillator, transfer matrices and **Q** operators in CFT possessing $U_q(\widehat{sl}(2))$.[9–12] Here the spectral determinants imply $D(E)=\prod_{E_j \in \text{eigenvalue}} (1-E/E_j)$ and its generalizations. A curious interplay between $D(E)$ and generalized Stokes multipliers is also found.[1, 10] In view of solvable models, a striking fact is that they share the same functional relations with transfer matrices in the fusion hierarchy possessing $U_q(\widehat{sl}(2))$.[10, 13–15] This allows for applications of the strong machineries in the study of solvable models[14–27] to the studies of Stokes multipliers, spectral determinants and so on. Several results have been explicitly obtained for the anharmonic oscillator problem, and are extended to higher differential analogues.[16, 17]

In this note, we consider an anharmonic oscillator perturbed by a lower power potential term. It belongs to a class of potentials discussed generally in refs. 6 and 7 with the exact resolution method. To be precise, we consider the eigenvalue problem,

$$\begin{aligned} \mathscr{H}(x,\alpha)\,\Psi_k(x) &= \left(-\frac{d^2}{dx^2}+x^{2M}+\alpha x^{M-1}\right)\Psi_k(x) \\ &= E_k\,\Psi_k(x) \end{aligned} \tag{1}$$

Throughout this report we set $\hbar=1$ and $M>1$.

The spectral problem concerning this Hamiltonian turns out to be in a category to which one can apply the tools in solvable models.

The sign of α seems to be crucial if one considers the operator (1) on the real axis.

We will not expect much difference from the "pure" anharmonic oscillator when $\alpha>0$, while we do expect change for $\alpha<0$ as the potential develops the double well.

It will be shown, however, that the negative α and the positive α problems are not separable when we discuss the global connection problem. Roughly speaking, the negative α problem is coupled to the positive α problem by crossing a border line of neighboring sectors and vice versa. See Section 2 for precise arguments. It may be then reasonable

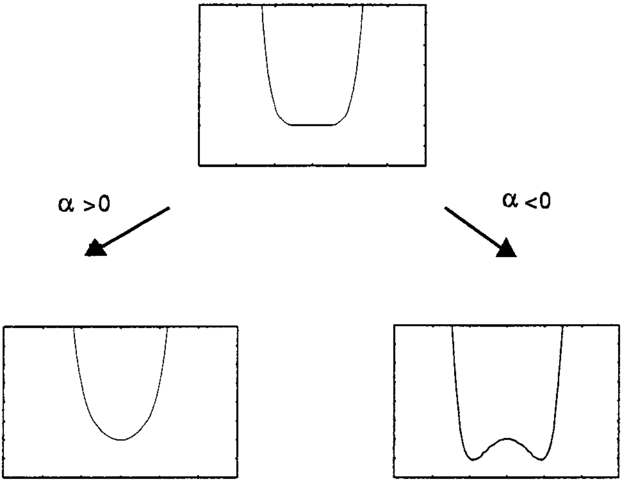

Fig. 1. An anharmonic oscillator perturbed by a positive or a negative perturbation term.

to consider a two-fold connection problem (crossing two adjacent lines), or more generally, relations between sectors separated by even multiples of border lines. Some of the Stokes multipliers, in the generalized connection problem, possess expressions corresponding to the eigenvalues of the (fusion) transfer matrices of the 3 state Perk–Schulz (PS) model[28–30] of which underlying symmetry is $U_q(\widehat{gl}(2\,|\,1))$. Others can not be directly equated with the (fusion) transfer matrices but have relations with the 3 state PS model as well. Thus we conclude that the perturbation αx^{M-1} breaks the $U_q(\widehat{sl}(2))$ symmetry of the "pure" anharmonic oscillator but it brings the new symmetry $U_q(\widehat{gl}(2\,|\,1))$. The deformation parameter q is related to the exponent of the perturbation by $q = \exp(i(\pi/(M+1)))$. Through these findings, we can derive the nonlinear integral equations (NLIE) which characterize the energy levels of both the negative α problem and the positive α problem simultaneously.

The paper is organized as follows. In the next section, we will explore symmetries of solutions to (1). The precise definition of sectors is given. The connection problem is addressed in Section 3. Certain components in fusion Stokes matrices are identified with eigenvalues of fusion transfer matrices associated to $U_q(\widehat{gl}(2\,|\,1))$. The spectral determinant is explicitly parameterized by FSS in a sector. The coupled NLIE are then derived in Section 4, which determine energy levels. We will perform analytical and numerical checks on the consistency of our result in Section 5. Section 6 is devoted to summary and discussions on open problems.

2. ASYMPTOTIC EXPANSION AND SYMMETRY OF SOLUTIONS

Let $\phi(x, \alpha, E)$ be an entire function of (x, α, E) and a solution to $\mathscr{H}(x, \alpha)\,\phi(x, \alpha, E) = E\phi(x, \alpha, E)$.

The solution, which decays exponentially at $x \to \infty$, is of primary interest. By employing the argument in ref. 31, we immediately find its asymptotic behavior,

$$\phi(x, \alpha, E) \sim x^{-M/2 - \alpha/2} \exp\left(-\frac{x^{M+1}}{M+1}\right) \tag{2}$$

$$\partial_x \phi(x, \alpha, E) \sim x^{M/2 - \alpha/2} \exp\left(-\frac{x^{M+1}}{M+1}\right) \tag{3}$$

The validity of the above expansion is not restricted to the real axis, but extends to the wedge, $|\arg x| < 3\pi/(2M+2)$.[13, 31]

The second order linear differential equation admits another independent solution. To specify it, or to deal with the global problem, it is convenient to extend x to the complex plane as mentioned in introduction. Then, as in the case of $\alpha = 0$, the solution exhibits a symmetry by rotating the complex x plane by a specific angle.

The direct calculation proves the following.

Theorem 1. Let $\phi(x, \alpha, E)$ be the above solution and $q = \exp(i(\pi/(M+1)))$. Then $\phi(q^{-1}x, q^{M+1}\alpha, q^2E)$ is also the solution to the differential equation, $\mathscr{H}(x, \alpha)\,\phi = E\phi$.

This is the desired second solution which grows exponentially on the positive real axis: $x^{-M/2 + \alpha/2} \exp(x^{M+1}/(M+1))$ for $x \to \infty$. We note that $q^{M+1}\alpha = -\alpha$. This deserves an attention. As mentioned in introduction, the potential assumes the completely different structure for α positive and α negative on the real axis. The rotation in the complex x plane by angle $\pi/(M+1)$, however, couples these two problems. Thus we shall treat the Hamiltonians with $\pm\alpha$ *simultaneously*. Similar pairing of differential equations is found for the positive and the negative angular momentum terms in a class of 3rd order differential equations.[16]

This observation is crucial in our approach and can be generalized further. To state it, we prepare some notations.

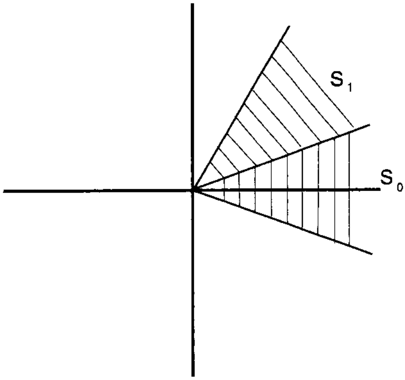

Fig. 2. The complex plane is divided into sectors. $\mathscr{S}_0$ and $\mathscr{S}_1$ are indicated as examples.

Hereafter α always takes a non-negative real value. By $\mathscr{H}^{(\varepsilon)}(x, \alpha)$, we mean the Schrödinger operator,

$$-\frac{d^2}{dx^2} + x^{2M} + \varepsilon\alpha x^{M-1}$$

where $\varepsilon = \pm 1$.

Let $\mathscr{S}_k$ be a sector in the plane satisfying

$$\left|\arg x - \frac{k\pi}{M+1}\right| \leqslant \frac{\pi}{2M+2}$$

The FSS depends on the sector. We conveniently define

$$y_j^{(\varepsilon)} := \frac{q^{j/2 - \varepsilon\alpha j/2}}{\sqrt{2i}} \phi(xq^{-j}, \varepsilon\alpha, q^{2j}E)$$

Theorem 2. For the $\mathscr{H}^{(\varepsilon)}(x, \alpha)$, the FSS in the sector $\mathscr{S}_j$ is given by $(y_j^{(\varepsilon_j)}, y_{j+1}^{(\varepsilon_{j+1})})$ where $\varepsilon_j = \varepsilon(-1)^j$.

For $\alpha = 0$ case, this has been argued in refs. 13 and 10. It is easily checked that $y_j^{(\varepsilon_j)}$ is the sub-dominant solution in $\mathscr{S}_j$; it tends to zero as x tends to infinity along in any direction in the sector.

In the next section, we consider the global connection problem of these FSS in the complex x plane.

3. FUSION STOKES MULTIPLIERS, $U_q(\widehat{gl}(2\,|\,1))$ STRUCTURE AND SPECTRAL DETERMINANTS

We introduce the Wronskian matrix

$$\Phi_j^{(\varepsilon)}(x) := \begin{pmatrix} y_j^{(\varepsilon)}, & y_{j+1}^{(-\varepsilon)} \\ \partial_x y_j^{(\varepsilon)}, & \partial_x y_{j+1}^{(-\varepsilon)} \end{pmatrix} \tag{4}$$

and the Wronskian $W_k^{(\varepsilon)} := \det \Phi_k^{(\varepsilon)}(x)$.

The linear dependence of the solution can be easily verified by evaluating the Wronskian at $\mathcal{S}_{j+1/2}$ using the asymptotic expansions (2) and (3). The present normalization yields $W_k^{(\varepsilon)} = 1$.

Let $\mathcal{M}_{j,1}^{(\varepsilon)}$ be the Stokes matrix connecting the Wronskian matrices $\Phi_j^{(\varepsilon)}(x)$ and $\Phi_{j+1}^{(-\varepsilon)}(x)$,

$$\Phi_j^{(\varepsilon)}(x) = \Phi_{j+1}^{(-\varepsilon)}(x)\, \mathcal{M}_{j,1}^{(\varepsilon)} \tag{5}$$

It permits an explicit parameterization

$$\mathcal{M}_{j,1}^{(\varepsilon)} := \begin{pmatrix} \tau_j^{(\varepsilon)}, & 1 \\ -1, & 0 \end{pmatrix} \tag{6}$$

where $\tau_j^{(\varepsilon)}$ is referred to as the Stokes multiplier. We have two remarks. First, the (1, 1) element is the function of α and $q^{2j}E$. We omit the dependency on α. The dependency on E is indicated by the index j. Second, the (2, 1) element, -1, is a consequence of the present normalization of the Wronskian.

To be more specific, we consider the operator $\mathcal{H}^{(+)}(x, \alpha)$ and start from the positive real axis (or more generally $\mathcal{S}_0$). The initial FSS is $(y_0^{(+)}, y_1^{(-)})$.

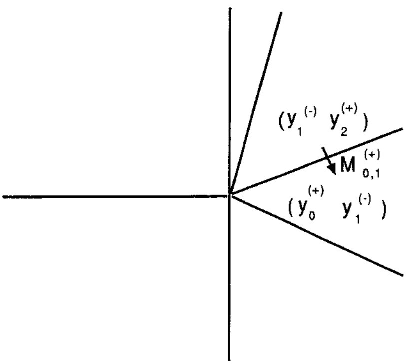

Fig. 3. The FSS in $\mathcal{S}_0$ and $\mathcal{S}_1$ are related by the matrix $\mathcal{M}_{0,1}^{(+)}$.

A linear relation follows from (5) between this FSS and $(y_1^{(-)}, y_2^{(+)})$, the FSS at the neighboring sector $\mathscr{S}_1$,

$$y_0^{(+)} = \tau_0^{(+)} y_1^{(-)} - y_2^{(+)} \tag{7}$$

Similarly, the FSS in $\mathscr{S}_2$ is linked to the FSS in $\mathscr{S}_1$ by

$$y_1^{(-)} = \tau_1^{(-)} y_2^{(+)} - y_3^{(-)} \tag{8}$$

Judging from the upper indices which indicate the corresponding signs of α, it may be natural to introduce a generalized Stokes matrix $\mathscr{M}_{0,2}^{(+)}$ connecting FSS $(y_0^{(+)}, y_1^{(-)})$ and $(y_2^{(+)}, y_3^{(-)})$. It is simply obtained by the matrix multiplication,

$$\mathscr{M}_{0,2}^{(+)} = \mathscr{M}_{1,1}^{(-)} \mathscr{M}_{0,1}^{(+)} = \begin{pmatrix} \tau_1^{(-)}\tau_0^{(+)} - 1, & \tau_1^{(-)} \\ -\tau_0^{(+)}, & -1 \end{pmatrix} \tag{9}$$

Equations (7) and (8) yield τ's in terms of y's. The (1, 1) component in (9), hereafter denoted by $T_{1,1}(E)$, is then represented in terms of y as,

$$T_{1,1}(E) = \tau_1^{(-)}\tau_0^{(+)} - 1 = \frac{y_0^{(+)}}{y_2^{(+)}} + \frac{y_0^{(+)} y_3^{(-)}}{y_2^{(+)} y_1^{(-)}} + \frac{y_3^{(-)}}{y_1^{(-)}} \tag{10}$$

The dependence of E in the rhs is implicitly indicated by indices of y. We will comment on this representation in terms of a solvable model later.

There is another expression using both y's and ∂y's. This form is of practical use in the following generalization. By applying the Cramer formula to (5), we immediately obtain

$$\tau_j^{(\varepsilon)} = \begin{vmatrix} y_j^{(\varepsilon)}, & y_{j+2}^{(\varepsilon)} \\ \partial_x y_j^{(\varepsilon)}, & \partial_x y_{j+2}^{(\varepsilon)} \end{vmatrix} \tag{11}$$

Note that we use the fact that the Wronskian is normalized to be unity. The (1, 1) entries in (9) is then given by

$$\begin{vmatrix} y_0^{(+)}, & y_3^{(-)} \\ \partial_x y_0^{(+)}, & \partial_x y_3^{(-)} \end{vmatrix}$$

One can further generalize the above result. Naturally, the "fusion" Stokes matrix $\mathscr{M}_{j,2k}^{(+)}$ is defined which relates FSS of $\mathscr{S}_j$ to $\mathscr{S}_{j+2k}$. Explicitly, it is given by

$$\mathscr{M}^{(+)}_{0,2k} = \begin{pmatrix} \begin{vmatrix} y_0^{(+)}, & y_{2k+1}^{(-)} \\ \partial_x y_0^{(+)}, & \partial_x y_{2k+1}^{(-)} \end{vmatrix}, & \begin{vmatrix} y_1^{(-)}, & y_{2k+1}^{(-)} \\ \partial_x y_1^{(-)}, & \partial_x y_{2k+1}^{(-)} \end{vmatrix} \\ -\begin{vmatrix} y_0^{(+)}, & y_{2k}^{(+)} \\ \partial_x y_0^{(+)}, & \partial_x y_{2k}^{(+)} \end{vmatrix}, & -\begin{vmatrix} y_1^{(-)}, & y_{2k}^{(+)} \\ \partial_x y_1^{(-)}, & \partial_x y_{2k}^{(+)} \end{vmatrix} \end{pmatrix} \tag{12}$$

for $j=0$. We can prove the above using the induction on k most easily. Similar formula holds for $\mathscr{M}^{(-)}_{0,2k}$ by replacing all upper indices $+ \leftrightarrow -$.

We are now ready to relate an entry in a fusion Stokes matrix to the spectral determinant. Hereafter we assume $M=2m-1$. It follows from the above argument that

$$\Phi^{(+)}_{2m} = \Phi^{(+)}_0 (\mathscr{M}^{(+)}_{0,2m})^{-1} \tag{13}$$

$y_0^{(+)}$ ($y_{2m}^{(+)}$) stands for the subdominant solution on the positive (negative) real axis. They tend to zero asymptotically in their proper region, being appropriate basis for the eigenfunction. The Eq. (13) tells, however, that $y_{2m}^{(+)}$ is combined to *both* $y_0^{(+)}$ *and* $y_1^{(-)}$ by rotating the complex plane by $-\pi$,

$$y_{2m}^{(+)} = (c_1 y_0^{(+)} + c_2 y_1^{(-)}) / \det \mathscr{M}^{(+)}_{0,2m}$$

$$c_1 = (\mathscr{M}^{(+)}_{0,2m})_{1,1}, \qquad c_2 = -(\mathscr{M}^{(+)}_{0,2m})_{2,1}$$

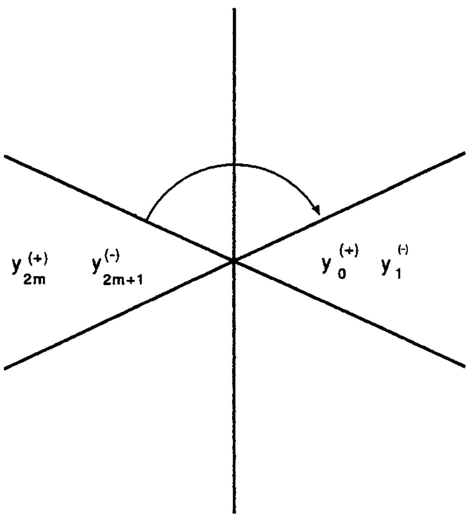

Fig. 4. The connection of FSS on the negative and the positive real axis is accomplished by $\mathscr{M}^{(+)}_{0,2m}$.

This is an obstacle in constructing an eigenfunction defined on the whole real axis. The prescription is to demand that the coefficient of $y_1^{(-)}$ must vanish if E is an eigenvalue. Consequently, it is proportional to the spectral determinant.

The coefficient is essentially equal to the (2, 1) component of $\mathscr{M}_{0,2m}^{(+)}$, and it reads in terms of the original ϕ function as

$$\frac{q^{\alpha(m+1)}}{2}\begin{vmatrix} \phi(x,\alpha,E), & \phi(-x,\alpha,E) \\ \partial_x\phi(x,\alpha,E), & \partial_x\phi(-x,\alpha,E) \end{vmatrix}$$

The x dependencies are spurious as the entities are products of Stokes multipliers which are obviously x independent. We adopt the simplest choice $x=0$. The coefficient is now proportional to $\phi(0,\alpha,E)\,\partial_x\phi(x,\alpha,E)|_{x=0}$. Thus we conclude that for an eigenvalue $E_j^{(+)}$ of $\mathscr{H}^{(+)}(x,\alpha)$,

$$\phi(0,\alpha,E_j^{(+)})=0 \qquad \text{or} \qquad \partial_x\phi(x,\alpha,E_j^{(+)})|_{x=0}=0 \tag{14}$$

must hold.

We can repeat the same argument starting from $\mathscr{H}^{(-)}(x,\alpha)$ on the positive real axis.

The above observation may lead to the identification

$$\phi(0,\varepsilon\alpha,E)\sim D_-^{(\varepsilon)}(E):=\prod_j\,(1-E/E_{-,j}^{(\varepsilon)}) \tag{15}$$

$$\partial_x\phi(x,\varepsilon\alpha,E)|_{x=0}\sim D_+^{(\varepsilon)}(E):=\prod_j\,(1-E/E_{+,j}^{(\varepsilon)}) \tag{16}$$

The lower sign signifies the parity: the positive parity means a contribution from symmetric wave function. The product must be taken over eigenvalues with the corresponding parity. The total set eigenvalues $\{E_j^{(\varepsilon)}\}$ of $\mathscr{H}^{(\varepsilon)}(x,\alpha)$ consists of two subsets, $\{E_j^{(\varepsilon)}\}=\{E_{+,j}^{(\varepsilon)}\}\cup\{E_{-,j}^{(\varepsilon)}\}$ and $D^{(\varepsilon)}(E)=D_+^{(\varepsilon)}(E)\,D_-^{(\varepsilon)}(E)$.

We comment on the relation of the present result to an existing solved model. $T_{1,1}(E)$ in (10) can be represented, utilizing (16), as

$$T_{1,1}(E)=q^{\alpha-1}\,\frac{D_-^{(+)}(E)}{D_-^{(+)}(q^4E)}+q^{2\alpha}\,\frac{D_-^{(+)}(E)\,D_-^{(-)}(q^6E)}{D_-^{(+)}(q^4E)\,D_-^{(-)}(q^2E)}+q^{\alpha+1}\,\frac{D_-^{(-)}(q^6E)}{D_-^{(-)}(q^2E)} \tag{17}$$

where we safely choose $x=0$ in the rhs.

The above expression has similarity to the dressed vacuum form (DVF) of the (unfused) transfer matrix for the 3-state PS model[30] with grading $(+, -, +)$. The latter can be found in Eq. (3.1) and (3.2) of ref. 32. The spectral parameter v corresponds to energy in the Schrödinger operator, precisely, $E = \exp(2\pi v/(M+1))$.

The spectral determinants have the following identification to the eigenvalues of Baxter's Q operators,

$$D_{-}^{(+)}(E^{q^{2j}}) = Q_2\left(v + \frac{j-2}{2}\, i\right)$$

$$D_{-}^{(-)}(E^{q^{2j}}) = Q_1\left(v + \frac{j-1}{2}\, i\right).$$

Those with positive parity may be identified with the second solutions of Baxter's Q operators.

The scalar factors (vacuum expectation values) $f_a(x)$, $g_a(x)$ in ref. 32 depend on the choice of the quantum space. We assume that the present quantum space space gives the simple scalars as in (17). In this sense, $T_{1,1}(E)$ exhibits the hidden $U_q(\widehat{gl}(2\,|\,1))$ symmetry behind the present Schrödinger operator, just as in the $U_q(\widehat{sl}(2))$ symmetry for $\alpha = 0$ problem. This coincidence can be observed further. We have checked up to certain value of k that the $(1, 1)$ element and the $(2, 2)$ element of $\mathscr{M}^{(+)}_{0,2k}$ coincide with DVF of symmetric fusion transfer matrices $\Lambda^{(1)}_{k}$ and $-\Lambda^{(1)}_{k-1}$ in ref. 32, respectively. The interpretation of the $(1, 2)$ and the $(2, 1)$ element, in terms of fusion transfer matrices, is still an open problem.

One can adopt another description of $T_{1,1}$. The $(2, 1)$ component of (5) results

$$\tau_0^{(\varepsilon)} = \frac{\partial y_0^{(\varepsilon)}}{\partial y_1^{(-\varepsilon)}} + \frac{\partial y_2^{(\varepsilon)}}{\partial y_1^{(-\varepsilon)}} \tag{18}$$

Proceeding as above, we arrive at,

$$T_{1,1}(E) = q^{\alpha+1}\,\frac{D_{+}^{(+)}(E)}{D_{+}^{(+)}(q^4E)} + q^{2\alpha}\,\frac{D_{+}^{(+)}(E)\,D_{+}^{(-)}(q^6E)}{D_{+}^{(+)}(q^4E)\,D_{+}^{(-)}(q^2E)} + q^{\alpha-1}\,\frac{D_{+}^{(-)}(q^6E)}{D_{+}^{(-)}(q^2E)} \tag{19}$$

In the next section, we determine the energy levels by utilizing the above results.

4. NONLINEAR INTEGRAL EQUATIONS FOR EIGENVALUE PROBLEM

The Bethe ansatz equations follow from the pole-free property of $T_{1,1}(E)$ on the real E axis,

$$q^{\alpha-1}\frac{D_{+}^{(-)}(q^{2}E_{+,j}^{(+)})}{D_{+}^{(-)}(q^{-2}E_{+,j}^{(+)})}=q^{-\alpha-1}\frac{D_{+}^{(+)}(q^{2}E_{+,j}^{(-)})}{D_{+}^{(+)}(q^{-2}E_{+,j}^{(-)})}=-1 \tag{20}$$

$$q^{\alpha+1}\frac{D_{-}^{(-)}(q^{2}E_{-,j}^{(+)})}{D_{-}^{(-)}(q^{-2}E_{-,j}^{(+)})}=q^{-\alpha+1}\frac{D_{-}^{(+)}(q^{2}E_{-,j}^{(-)})}{D_{-}^{(+)}(q^{-2}E_{-,j}^{(-)})}=-1 \tag{21}$$

To their analysis, we apply the strong machinery in solvable models, the method of nonlinear integral equations. Hereafter we shall confine ourselves to the case $0\leqslant\alpha\leqslant M$ where energies are nonnegative. The simple pattern of energy spectrum permits the following simple-minded choice of auxiliary functions,

$$a_{\varepsilon'}^{(\varepsilon)}(E):=q^{\varepsilon\alpha-\varepsilon'}\frac{D_{\varepsilon'}^{(-\varepsilon)}(q^{2}E)}{D_{\varepsilon'}^{(-\varepsilon)}(q^{-2}E)} \tag{22}$$

$$A_{\varepsilon'}^{(\varepsilon)}(E):=1+a_{\varepsilon'}^{(\varepsilon)}(E) \tag{23}$$

Thus

$$A_{\varepsilon'}^{(\varepsilon)}(E_{\varepsilon',j}^{(\varepsilon)})=0 \tag{24}$$

Remember that $\varepsilon(=\pm1)$ represents the signature of the perturbation while $\varepsilon'(=\pm1)$ denotes the parity.

In addition, we need some inputs about the asymptotic behaviors from the WKB method. Fortunately, they are already available[9] as the existence of lower power term does not alter them.

$$\ln D_{\pm}^{(\varepsilon)}(E)\sim\frac{a_0}{2}(-E)^{\mu},\qquad |E|\to\infty,\quad |\arg(-E)|<\pi \tag{25}$$

$$b_0(E_j^{(\varepsilon)})^{\mu}\sim 2\pi\left(j+\frac{1}{2}\right),\qquad j\to\infty \tag{26}$$

$$\mu=\frac{M+1}{2M},\qquad a_0=\frac{b_0}{2\sin\mu\pi},\qquad b_0=\frac{\pi^{1/2}\Gamma(1/2M)}{M\Gamma(3/2+1/2M)}$$

There might be several routes to reach nonlinear integral equations among $a^{(\varepsilon)}_{(\varepsilon')}$ and $A^{(\varepsilon)}_{(\varepsilon')}$. Here we choose the quickest way[9, 16, 21, 23] which fully exploits the fact that zeros of $D^{(\varepsilon)}_{\pm}(E)$ are on the positive real θ axis. In addition, we assume that there are no zeros of $A^{(\varepsilon)}_{\pm}(E)$ inside the narrow strip including the positive real axis other than those from zeros of $D^{(\varepsilon)}_{\pm}(E)$.

Clearly we have

$$\log a^{(\varepsilon)}_{\varepsilon'}(E) = \frac{(\varepsilon\alpha - \varepsilon')\,\pi}{M+1}\, i + \sum_j F\left(\frac{E}{E^{(-\varepsilon)}_{\varepsilon', j}}\right)$$

$$F(E) = \log \frac{1 - q^2 E}{1 - q^{-2} E}$$

The above assumption allows the representation of the summation part by an integral over contour $\mathscr{C}_E$ which surrounds the positive real axis counterclockwise,

$$\log a^{(\varepsilon)}_{\varepsilon'}(E) = \frac{(\varepsilon\alpha - \varepsilon')\,\pi}{M+1}\, i + \frac{1}{2\pi i} \int_{\mathscr{C}_E} dE' F\left(\frac{E}{E'}\right) \partial_{E'} \log A^{(-\varepsilon)}_{\varepsilon'}(E') \tag{27}$$

For convenience, we introduce a variable θ[9] by

$$E = \exp(\theta/\mu)/v^2 \qquad v = (2M+2)^{-1/2\mu} \Big/ \Gamma\left(\frac{1}{2\mu}\right)$$

which originates from the matching condition of the WKB result (25) and the Q-operator analysis.[23, 33]

Let $\mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta)$, $\mathfrak{A}^{(\varepsilon)}_{\varepsilon'}(\theta)$ be auxiliary functions defined in (22), (23) regarded as functions of θ. Then Eq. (27) reads,

$$\log \mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta) = \frac{(\varepsilon\alpha - \varepsilon')\,\pi}{M+1}\, i + \frac{1}{2\pi i} \int_{\mathscr{C}_\theta} d\theta'\, G(\theta - \theta')\, \partial_{\theta'} \log \mathfrak{A}^{(-\varepsilon)}_{\varepsilon'}(\theta')$$

$$G(\theta) = \log\left(q^2 \frac{\sinh(M\theta/(M+1) + i(\pi/(M+1)))}{\sinh(M\theta/(M+1) - i(\pi/(M+1)))}\right)$$

where $\mathscr{C}_\theta$ encircles the whole real axis counterclockwise.

For the reason which will be supplemented, we shall keep $\mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta)$ in the lower half plane but use $1/\mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta)$ in the upper half plane.

This requirement modifies the above expression as

$$\log \mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta) = \frac{(\varepsilon\alpha - \varepsilon')\,\pi}{M+1}\, i - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\theta^1\, \partial_\theta G(\theta - \theta' + i0)\, \log \mathfrak{a}^{(-\varepsilon)}_{\varepsilon'}(\theta' - i0)$$
$$+ \frac{1}{\pi}\, \Im\left(\int_{-\infty}^{\infty} d\theta'\, \partial_\theta G(\theta - \theta' + i0)\, \log \mathfrak{A}^{(-\varepsilon)}_{\varepsilon'}(\theta' - i0)\right) \tag{28}$$

where θ is assumed to possess small negative imaginary part. The property $(a^{(\varepsilon)}_{\varepsilon'}(\theta))^* = 1/a^{(\varepsilon)}_{\varepsilon'}(\theta^*)$ is employed in the above transformation.

We solve (28) in terms of $\log \mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta)$ to reach the final expression of NLIE,

$$\ln \mathfrak{a}^{(\varepsilon)}_{\varepsilon'}(\theta) = -\frac{i}{2}\, b_0 \nu^{-2\mu} \mathrm{e}^{\theta} + \frac{\pi}{2}\, i\left(-\varepsilon' + \varepsilon\,\frac{\alpha}{M}\right)$$
$$+ 2i\Im\left\{\int_{-\infty}^{\infty} K_1(\theta - \theta' + i0)\, \ln \mathfrak{A}^{(\varepsilon)}_{\varepsilon'}(\theta' - i0)\, d\theta'\right.$$
$$\left. + \int_{-\infty}^{\infty} K_2(\theta - \theta' + i0)\, \ln \mathfrak{A}^{(-\varepsilon)}_{\varepsilon'}(\theta' - i0)\, d\theta'\right\} \tag{29}$$

The kernel functions read

$$K_1(\theta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\theta}\, \frac{\sinh^2((\pi(M-1)\, w)/2M)}{\sinh \pi w\, \sinh(\pi w/M)}\, dw$$

$$K_2(\theta) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iw\theta}\, \frac{\sinh((\pi(M+1)\, w)/2M)\, \sinh((\pi(M-1)\, w)/2M)}{\sinh \pi w\, \sinh(\pi w/M)}\, dw.$$

Few remarks are in order.

1. As a consequence of connection rules, the integral equations are coupled for auxiliary functions related to the positive and the negative coefficient of x^{M-1}.

2. On the other hand, equations with different parities are decoupled.

3. The constants are determined from the consistency by putting $\theta \to -\infty$. The α dependence is only summarized in these constants.

4. The first term in the rhs is determined so that we recover the result from the WKB method (25) by dropping contributions of integrals.

Clearly, $\mathfrak{a}_{\varepsilon'}^{(\varepsilon)}(\theta)$ is bounded in the upper-half plane. This explains our choice of appropriate half planes for auxiliary functions.

The eigenvalues $\{E_{j,\pm}^{(\varepsilon)}\}$ are evaluated by

$$\ln \mathfrak{a}_{\pm}^{(\varepsilon)}(\theta_{j,\pm}^{(\varepsilon)}) = (2j+1)\,\pi i \qquad \text{and} \qquad E_{j,\pm}^{(\varepsilon)} = \exp(\theta_{j,\pm}^{(\varepsilon)}/\mu)/v^2$$

More explicitly

$$\frac{1}{2}\, b_0 v^{-2\mu} e^{\theta_{j,\varepsilon'}^{(\varepsilon)}} = \left(2j+1-\varepsilon'\,\frac{1}{2}+\varepsilon\,\frac{\alpha}{2M}\right)\pi$$
$$+2\Im\left\{\int_{-\infty}^{\infty} K_1(\theta_{j,\varepsilon'}^{(\varepsilon)}-\theta'+i0)\ln \mathfrak{A}_{\varepsilon'}^{(\varepsilon)}(\theta')\,d\theta'\right.$$
$$\left.+\int_{-\infty}^{\infty} K_2(\theta_{j,\varepsilon'}^{(\varepsilon)}-\theta'+i0)\ln \mathfrak{A}_{\varepsilon'}^{(-\varepsilon)}(\theta')\,d\theta'\right\} \qquad (j \geqslant 0) \qquad (30)$$

We present examples of numerical solutions to (28) in Fig. 5. The real and the imaginary parts of $\ln \mathfrak{A}_{+}^{(\pm)}$ are depicted for $M=3$, $\alpha=1.5$.

5. BENCHMARKS

We shall check the nonlinear integral equations analytically for limiting cases and numerically.

(1) $\alpha=0$ case

By putting, $\mathfrak{a}_{\pm}^{(+)}(E)=\mathfrak{a}_{\pm}^{(-)}(E)$ the coupled NLIE reduce to an identical integral equation. Immediately seen, the result coincides with the nonlinear integral equation in ref. 9.

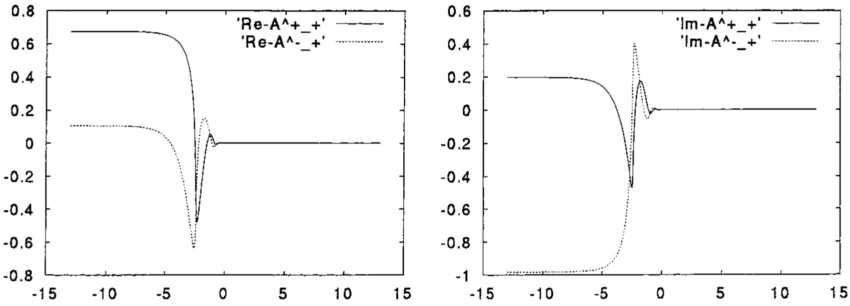

Fig. 5. Left: the real part of $\ln \mathfrak{A}_{+}^{(\pm)}$, Right: the imaginary part of $\ln \mathfrak{A}_{+}^{(\pm)}$.

(2) $\alpha = M$ case

In this case, we have a duality in energy spectra; $\{E_j^{(+)}\}$ coincide with $\{E_j^{(-)}\}$, except for $E_0^{(-)} = 0$ in the latter. This degeneracy can be easily explained by the following representation of the Hamiltonians,[34, 35]

$$\mathscr{H}^{(-)}(x, \alpha = M) = \mathscr{D}^\dagger \mathscr{D}$$

$$\mathscr{H}^{(+)}(x, \alpha = M) = \mathscr{D}\mathscr{D}^\dagger$$

$$\mathscr{D} = \frac{1}{i}\frac{d}{dx} - ix^M$$

Once an eigenvector $\mathscr{H}^{(-)}(x, M)\,\psi_j^{(-)} = E_j^{(-)}\psi_j^{(-)}$ is found, we can construct the eigenvector for $\mathscr{H}^{(+)}(x, \alpha)$ with the same energy by $\psi_{j-1}^{(+)} := \mathscr{D}\psi_j^{(-)}$. Only the exception is the $j = 0$ case where $\mathscr{D}\psi_{j=0}^{(-)} = 0$. It is interesting that the asymptotic form (2) from the WKB type argument is exact for all x in this case.[3]

The above facts can be also verified from (29). Note that the rhs can be treated as the mod $2\pi i$ quantity. Then the choice $\alpha = M$ leads to the same coupled equations under identifications $\mathfrak{a}_+^{(+)}(\theta) \leftrightarrow \mathfrak{a}_-^{(-)}(\theta)$, $\mathfrak{a}_-^{(+)}(\theta) \leftrightarrow \mathfrak{a}_+^{(-)}(\theta)$. This explains the degeneracy of the spectra as it consists both from the negative and the positive parity contributions. The zero energy case must be treated more separately. By choosing $j = 0$, $\varepsilon = -\varepsilon' = -1$, $\alpha = M$, we find the first term in lhs of (30) is null. So the first order approximation is $\theta_{+,0}^{(-)} = -\infty$. Actually this is exact as we determine the constant terms so that NLIE are consistent in $\theta \to -\infty$. See Remark 3 after (29). This solution gives the missing energy 0.

Finally we present the preliminary numerical results for $M = 3$.

Table I shows the results from the IMSL package (dsleig.f), the (naive) WKB method and those obtained by solving the nonlinear equations. The agreement is not yet precise enough (typically 3–4 digits). Some implement is still in need for the numerical accuracy. Nevertheless, the NLIE data already show much improvement from the (naive) WKB results.

By the (naive) WKB method, we mean a self-consistent determination of $E_j^{(\varepsilon)}$ by

$$\oint |p|\, dx = \int_{-x_0}^{x_0} \sqrt{E_j^{(\varepsilon)} - x^6 - \varepsilon\alpha x^2}\, dx = \left(j + \frac{1}{2}\right)\pi \tag{31}$$

[3] I thank V. V. Bazhanov, R. J. Baxter, and B. Nienhuis for pointing out the explicit eigenfunction for $\psi_0^{(-)}$ right after my talk.

Table I. First Two Energy Levels Calculated by the IMSL Library, the (Naive) WKB Method and the NLIE Method. We Choose $M=3$ and Adopt Various α. See Asterisk and ◇ for the Text

α	IMSL 0th	IMSL 1st	WKB 0th	WKB 1st	NLIE 0th	NLIE 1st
−2.5000	0.22909	2.3741	◇	2.36641	0.22911	2.37385
−2.0000	0.44007	2.7962	0*	2.73228	0.44009	2.7959
−1.5000	0.63524	3.2028	0.17736	3.09594	0.63527	3.2025
−1.0000	0.81664	3.5949	0.38490	3.45603	0.81667	3.59506
−0.50000	0.98599	3.9732	0.59582	3.81142	0.98603	3.97303
0.0000	1.1448	4.3385	0.8008	4.16123	1.1448	4.3382
0.50000	1.2943	4.6917	0.99516	4.50476	1.29436	4.6918
1.0000	1.4356	5.0333	1.1768	4.84147	1.43569	5.0336
1.5000	1.5696	5.3642	1.3456	5.17101	1.5698	5.3640
2.0000	1.6972	5.6850	1.5024	5.49313	1.6973	5.68459
2.5000	1.8189	5.9962	1.6487	5.80773	1.8192	5.99597

where $E_j^{(\varepsilon)} - x_0^6 - \varepsilon\alpha x_0^2 = 0$. Particularly, for the value with asterisk, this method has subtlety. Immediately seen, $E_0^{(\varepsilon)} = 0$, $x_0 = 2^{1/4}$ is a formal solution to (31) for $j=0$, $\varepsilon = -$, $\alpha = 2$. It however involves an isolated turning point of the 2nd order at the origin if $E_0^{(\varepsilon)} = 0$, which spoils the simple application of the condition (31). The value with ◇ has similar difficultly. We however skip further discussion on the validity on the (naive) WKB method as it is out of the present subject.

Summarizing, we check the consistency of (29) in some limiting cases and by numerical methods.

6. SUMMARY AND DISCUSSION

In this report, the eigenvalue problem has been addressed for the 1D quantum systems of which Hamiltonians include double well potentials. We have successfully derived the coupled NLIE which determine energy levels of the systems with potential terms of $\pm\alpha x^{M-1} + x^{2M}$ at the same time.

The essence of our strategy is to utilize the following correspondences between 1D quantum mechanics and $1+1$D solvable models,

$$\text{energy} \Leftrightarrow \text{spectral parameter}$$

$$\text{Stokes multipliers} \Leftrightarrow \text{transfer matrices}$$

$$\text{eigenfunctions or derivatives at } x=0$$

$$\Leftrightarrow \text{vacuum expectation values of } Q \text{ operators}$$

We are then entitled to apply the strong machinery of the latter developed since Baxter's revolution.

There are several open questions.

1. In this report we confine ourselves to the simplest case $\alpha \leqslant M$. For $\alpha > M$, the existence of negative eigenvalues ruins the analyticity assumptions on auxiliary functions. Still, formal expressions of NLIE are possible which are similar to excited states TBA equations. The integration contour is, however, not so simple as described here. This is an apparent drawback in actual numerical investigations. The clever choice of auxiliary functions may be desired.

2. The understanding is lacking on the intrinsic reason why affine symmetry like $U_q(\widehat{sl}(2))$ or $U_q(\widehat{gl}(2 \mid 1))$ comes into play in this simple 1D quantum mechanical model.

3. This is somewhat related to the above, but is the most intriguing question. Where is the Yang–Baxter equation in the 1D Schrödinger operator problem? Once this is known, the fusion hierarchy, useful in the present study, is a mere corollary of it.

We hope to answer these in the future publication.

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